Robust Eigenstructure Assignment with Structured State Space Uncertainty

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Recent sufficient conditions for robust stability and robust performance of linear time-invariant systems subject to structured state-space uncertainty are utilized to obtain a robust eigenstructure assignment design method. This new approach optimizes either the sufficient condition for stability or performance robustness while constraining the dominant eigenvalues to lie within chosen regions in the complex plane. This constrained optimization problem is solved by using the sequential unconstrained minimization technique with a quadratic extended interior penalty function. The use of constraints on certain eigenvector entries and the effect of these constraints on robustness and nominal performance are considered. Conservatism of the robustness conditions is reduced by simultaneously introducing a similarity transformation, a positive real diagonal weighting, and a unitary weighting into the design procedure. An example that illustrates the design of a robust eigenstructure assignment controller for a pitch pointing/vertical translation maneuver of the AFTI F-16 aircraft is presented.

Introduction

EIGENSTRUCTURE assignment is a method that allows the incorporation of classical specifications on damping, settling time, and mode decoupling into a modern multivariable control framework. This approach was utilized by Sobel and Shapiro[1] to design a pitch pointing/vertical translation control law for the linearized longitudinal dynamics of the advanced fighter technology integration (AFTI) F-16 aircraft. This design achieved excellent classical performance specifications at the nominal flight condition. However, the design approach did not consider that the aircraft parameters, which consist of the stability derivatives and control derivatives, are uncertain.

Recently, Sobel et al.[2] have proposed a sufficient condition for the robust stability of a linear time-invariant system subject to linear time-varying structured state-space uncertainty. This result, which is based on the Gronwall lemma, ensures robust stability if the nominal eigenvalues lie to the left of a vertical line in the complex plane that is determined by a norm involving the structure of the uncertainty and the nominal closed-loop eigenvector matrix. An extension to performance robustness was proposed by Yu and Sobel[3] for linear time-invariant systems subject to linear time-invariant structured state-space uncertainty. The closed-loop eigenvalues are guaranteed to lie within chosen performance regions provided that the eigenvalues of a transformed system matrix lie to the left of a vertical line in the complex plane.

In this paper, we propose a robust eigenstructure assignment design method that optimizes either the sufficient condition for stability or performance robustness while constraining the dominant eigenvalues to lie within chosen performance regions in the complex plane. This constrained optimization problem is solved by using the sequential unconstrained minimization technique with a quadratic extended interior penalty function.[4] An example that illustrates the design of a robust eigenstructure assignment controller for a pitch pointing/vertical translation maneuver of the AFTI F-16 aircraft is presented. Several designs are shown, including 1) the nonrobust design of Ref. 1, 2) a robust design using a state transformation to reduce conservatism in the sufficient condition, 3) a robust design that includes explicit constraints on certain eigenvector entries to illustrate the tradeoff between robustness and nominal performance, 4) a robust design that utilizes both a state transformation and a unitary weighting matrix to reduce conservatism, and 5) a performance robust design that ensures that the closed-loop eigenvalues lie within a chosen damping-settling time region for all uncertainty. Time responses of the different designs to unit step commands in both flight-path angle and pitch attitude are shown to illustrate the penalty in nominal performance that results when using the robust design method.

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Robustness Results

Consider a nominal linear time-invariant multi-input/multi-output system described by

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx \tag{1b}$$

where $x \in R^n$ is the state vector, $u \in R^m$ the input vector, $y \in R^r$ the output vector, and A, B, and C the constant matrices.

Suppose that the nominal system is subject to linear time-varying uncertainties in the entries of A and B described by $\Delta A(t)$ and $\Delta B(t)$, respectively. We shall assume that the entries of $\Delta A(t)$ and $\Delta B(t)$ are continuous functions of time. Then, the system with uncertainty is given by

$$\dot{x} = Ax + Bu + \Delta A(t)x + \Delta B(t)u \tag{2a}$$

$$y = Cx \tag{2b}$$

Furthermore, suppose that bounds are available on the absolute values of the maximum variations in the elements of $\Delta A(t)$ and $\Delta B(t)$. That is,

$$|\Delta a_{ij}(t)| \le (a_{ij})_{\max}; \qquad i = 1, \ldots, n, \quad j = 1, \ldots, n \tag{3a}$$

$$|\Delta b_{ij}(t)| \le (b_{ij})_{\max}; \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m \tag{3b}$$

Define $\Delta A^+(t)$ and $\Delta B^+(t)$ as the matrices obtained by replacing the entries of $\Delta A(t)$ and $\Delta B(t)$ by their absolute values. In addition, define A_{max} and B_{max} as the matrices with entries $(a_{ij})_{max}$ and $(b_{ij})_{max}$, respectively. Then,

$$\{\Delta A(t)\colon \Delta A^{+}(t) \le A_{max}\} \tag{4a}$$

and

$$\{\Delta B(t)\colon \Delta B^{+}(t) \le B_{max}\} \tag{4b}$$

where "$\le$" is applied element by element to matrices and $A_{max} \in R_{+}^{n\times n}$ and $B_{max} \in R_{+}^{n\times m}$, where R_{+} is the set of non-negative numbers.

Consider the constant gain output feedback control law described by

$$u(t) = Fy(t) \tag{5}$$

Then, the nominal closed-loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) \tag{6}$$

and the uncertain closed-loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) + [\Delta A(t) + \Delta BFC(t)]x(t) \tag{7}$$

Stability Robustness Problem. Given a feedback gain matrix $F \in R^{m\times r}$ such that the nominal closed-loop system exhibits desirable dynamic performance, determine if the uncertain closed-loop system is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by Eqs. (4).

Performance Robustness Problem. A feedback gain matrix $F \in R^{m\times r}$ is chosen such that all of the eigenvalues of the nominal closed-loop system are inside the region R. Determine if all of the eigenvalues of the uncertain closed-loop system are inside the region R for all time-invariant ΔA and ΔB described by Eqs. (4).

The solution proposed by Sobel et al.[2] for the stability robustness problem is described by the following theorem.

Theorem. Suppose that F is such that the nominal closed-loop system described by Eq. (6) is asymptotically stable with distinct eigenvalues. Then, the uncertain closed-loop system given by Eq. (7) is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by Eqs. (4), if

$$\alpha > \pi\{(M^{-1})^{+}[A_{max} + B_{max}(FC)^{+}]M^{+}\} \tag{8}$$

where

$$\alpha = -\max_i \mathrm{Re}[\lambda_i(A + BFC)]$$

M is the modal matrix of $(A + BFC)$, and $\pi(\cdot)$ denotes the Perron eigenvalue. The Perron eigenvalue[5] of a non-negative matrix is the real non-negative eigenvalue $\lambda_{max} \ge 0$, such that $\lambda_{max} \ge |\lambda_i|$ for all eigenvalues of the non-negative matrix.

The preceding theorem describes a sufficient condition for the robust stability in terms of the eigenstructure of the nominal closed-loop system. Robust stability is ensured provided that the nominal closed-loop eigenvalues lie to the left of a vertical line in the complex plane that is determined by a norm involving the structure of the uncertainty and the nominal closed-loop modal matrix.

The solution proposed by Yu and Sobel[3] for the performance robustness problem is described by the following theorem.

Theorem. Suppose that F is such that the nominal closed-loop system described by Eq. (6) has only distinct eigenvalues, all of which lie inside the region R, which is denoted by the region to the left of the line L in Fig. 1. Let M be the modal matrix of $A + BFC$, let D be a diagonal matrix with positive real entries, and let Q be a nonsingular matrix. Then, the eigenvalues of the uncertain closed-loop system with time-invariant uncertainty described by

$$\dot{x}(t) = (A + BFC)x(t) + [\Delta A + \Delta BFC]x(t) \tag{9}$$

will be in R for all time-invariant ΔA and ΔB described by Eqs. (4) if

$$\min[\alpha_1,\alpha_2] > \kappa_2(Q)\cdot \|[(MDQ)^{-1}]^{+} \times [A_{max} + B_{max}(FC)^{+}][MDQ]^{+}\|_2 \tag{10}$$

where

$$\alpha_1 = -\max_i [\mathrm{Re}\lambda_i] + a$$

$$\alpha_2 = -\max_i \mathrm{Re}[e^{-j\theta}\lambda_i]$$

and $\kappa_2(Q) = \|Q\|_2 \cdot \|Q^{-1}\|_2$ is the 2-norm condition number of the matrix Q. In the expressions for α_1 and α_2, the quantity λ_i is an eigenvalue of $(A + BFC)$ and a is the value at which the curve L in Fig. 1 intersects the negative real axis. The region R in the preceding theorem ensures that a system with a dominant pair of complex conjugate eigenvalues has a damping ratio of $\zeta \ge \zeta_{min}$ and a settling time of $t_s \le (t_s)_{max}$.

Eigenstructure Assignment

Consider the linear time-invariant system described by Eqs. (1). We shall assume that matrices B and C have full rank. With these assumptions, the constant gain output feedback problem using eigenstructure assignment can be stated as follows: Given a set of desired eigenvalues $\{\lambda_i^d\}$, $i = 1, \ldots, r$, and a corresponding set of desired eigenvectors $\{v_i^d\}$, $i = 1, \ldots, r$, find a real $m \times r$ matrix F such that the eigenvalues of $A + BFC$ contain $\{\lambda_i^d\}$ as a subset and the corresponding eigenvectors of $A + BFC$ are close to the respective members of the set $\{v_i^d\}$. The following theorem, from Srinathkumar,[6] describes the number of eigenvalues and eigenvector entries that can be exactly assigned.

Theorem. Given the controllable and observable system described by Eqs. (1) and the assumptions that matrices B and C are full rank, then $\max(m,r)$ closed-loop eigenvalues can be assigned and $\max(m,r)$ eigenvectors can be partially assigned with $\min(m,r)$ entries in each eigenvector arbitrarily chosen using constant gain output feedback.

In general, we may desire to exercise some control over more than $\min(m,r)$ entries in a particular eigenvector. Therefore, we shall now discuss the problem of first characterizing desired eigenvectors v_i^d that can be assigned as closed-loop eigenvectors and then determining the best possible set of achievable eigenvectors in case desired eigenvector v_i^d is not achievable.

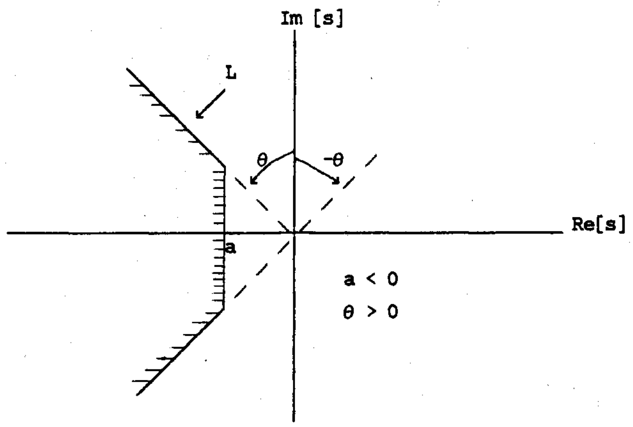

Fig. 1 Performance robustness region.

These problems were considered by Andry et al.,[7] who have shown the need for the eigenvector v_i to be in the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$. An alternative representation is described by Kautsky et al.,[8] who show that the subspace in which the eigenvector v_i must reside is also given by the null space of $U_1^T(\lambda_i I - A)$. The matrix U_1 is obtained from the singular-value decomposition of B, which is given by

$$B = [U_0, U_1]\begin{bmatrix} \Sigma V^T \\ 0 \end{bmatrix} \tag{11}$$

In general, however, a desired eigenvector v_i^d will not reside in the prescribed subspace and, hence, cannot be achieved. Instead, a "best possible" choice for an achievable eigenvector is made. It is suggested in Ref. 7 that this best possible eigenvector is the projection of v_i^d onto the subspace, which is the null space of $U_1^T(\lambda_i I - A)$. We remark that the desired eigenvectors v_i^d are chosen based on mode decoupling specifications. By using an orthogonal projection to obtain the achievable eigenvectors v_i^a, we are minimizing the 2-norm error between a desired eigenvector and its corresponding achievable eigenvector. It is important to note that the concept of stability robustness was not considered in the design approach proposed by Andry et al.[7]

Sequential Unconstrained Minimization Technique

The constrained optimization to be considered[4] is to find a vector p^* of design variables that minimizes either the function

$$J_1(p) = \pi\{(M^{-1})^+[A_{\max} + B_{\max}(FC)^+]M^+\} - \alpha \tag{12}$$

or the function

$$J_2(p) = \kappa_2(Q)\cdot \|[(MDQ)^{-1}]^+[A_{\max} + B_{\max}(FC)^+] \times [MDQ]^+\|_2 - \min(\alpha_1,\alpha_2) \tag{13}$$

subject to the constraints

$$g_i(p) \geq 0, \qquad i = 1, \ldots, q \tag{14}$$

The vector p of design variables will contain a subset of the real and imaginary parts of the closed-loop eigenvalues and a subset of the eigenvector parameters. To define the phrase "eigenvector parameters," we let L_i be a matrix whose columns form a basis for the null space of $U_1^T(\lambda_i I - A)$. Then, the ith achievable eigenvector v_i^a is given by

$$v_i^a = L_i z_i, \qquad i = 1, 2, \ldots, n \tag{15}$$

where the vector z_i contains the free design parameters. Thus, we define vector z_i, $i = 1, 2, \ldots, n$, to be the eigenvector parameters. Constraints $g_i(p)$ will be chosen to constrain some of the closed-loop eigenvalues to lie in desired regions in the complex plane. Constraints may also be placed on certain entries of the closed-loop eigenvectors to obtain a desired amount of mode decoupling. We remark that eigenvector constraints may be necessary to improve mode decoupling because a subset of the eigenvector parameters is included in the parameter vector p. Thus, the optimization will change this subset of eigenvector parameters without attempting to minimize the 2-norm difference between a desired eigenvector and its corresponding achievable eigenvector.

It is assumed that at least one of the constraints of Eq. (14) is critical at the minimum p^*; i.e., $g_i(p) = 0$ for some i. This constrained optimization problem may be transformed into a series of unconstrained minimization problems by introducing a penalty function associated with the constraints, and the transformed problem can be solved by the sequential unconstrained minimization technique.[4] The resulting transformed problem is to find the minimum of the function $P_k(\epsilon)$ as ϵ goes to zero, where

$$P_k(\epsilon) = J_k(p) + \epsilon\sum_{i=1}^{q} f_i(p), \qquad k = 1,2 \tag{16}$$

with $J_k(p)$ defined by either Eq. (12) or (13) and $f_i(p)$ defined by

$$f_i(p) = 1/g_i(p) \quad \text{if} \quad g_i(p) > 0 \tag{17}$$

The term $\epsilon f_i(p)$ represents the penalty associated with the ith constraint and is an interior penalty function in the sense that it is defined only if p is inside the feasible design domain. With p^ϵ denoting the point in the design space where $P_k(\epsilon)$ attains its minimum value for a given value of ϵ, it may be shown[4] that as ϵ goes to zero

$$\min P_k(\epsilon) \to J_k(p^*)$$

$$p^\epsilon \to p^*$$

Next, a quadratic extended interior penalty function[4] is introduced to permit the use of design points that are outside the feasible domain. The definition of f_i in Eq. (16) for the quadratic extended penalty function is[4]

$$f_i = \begin{cases} 1/g_i & \text{if} \quad g_i \geq g_0 \\ (1/g_0)[(g_i/g_0)^2 - 3(g_i/g_0) + 3] & \text{if} \quad g_i \leq g_0 \end{cases} \tag{18}$$

where

$$g_0 = \sqrt{\epsilon} \tag{19}$$

AFTI F-16 Pitch Pointing/Vertical Translation Design

Consider the linear time-invariant (LTI) model of the AFTI F-16 aircraft described by

$$\begin{bmatrix} \dot\gamma \\ \dot q \\ \dot\alpha \\ \dot\delta_e \\ \dot\delta_f \end{bmatrix} = \begin{bmatrix} 0 & 0.00665 & 1.3411 & 0.16897 & 0.25183 \\ 0 & -0.86939 & 43.223 & -17.251 & -1.5766 \\ 0 & 0.99335 & -1.3411 & -0.16897 & -0.25183 \\ 0 & 0 & 0 & -20.0 & 0 \\ 0 & 0 & 0 & 0 & -20.0 \end{bmatrix} \times \begin{bmatrix} \gamma \\ q \\ \alpha \\ \delta_e \\ \delta_f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \delta_{e_c} \\ \delta_{f_c} \end{bmatrix} \tag{20}$$

A pitch pointing/vertical translation controller was proposed by Sobel and Shapiro[1] to decouple the pitch-attitude and flight-path responses. The zero entries in the short-period eigenvectors are for pitch pointing (θ command with no coupling to γ), whereas the zero entry in the gamma mode eigenvector is for vertical translation (γ command with no coupling to θ or q). The desired eigenvectors are shown in Table 1, where we observe the zero entries that are chosen to obtain the required decoupling. The feedback gains described by Sobel

Table 1 Desired eigenvectors

Short period		Gamma mode	Actuator mode	Actuator mode	
0	0	1	x	x	γ
1	x	0	x	x	q
x	1	x	x	x	α
x	x	x	1	x	δ_e
x	x	x	x	1	δ_f

Table 2 Closed-loop eigenvalues and control gains

	Closed-loop eigenvalues	Feedforward gains		Feedback gains ($u = -Fx$)				
		γ_c	θ_c	γ	q	α	δ_e	δ_f
Nonrobust design	$-5.609 \pm j4.19$ -1.00 -19.0 -19.5	-0.375 4.12	-2.87 1.98	-3.25 6.10	-0.891 0.898	-7.11 10.02	0.526 -0.420	0.0840 -0.102
Robust design	$-3.629 \pm j5.18$ -2.66 -19.0 -19.5	-0.992 10.93	-1.62 -6.47	-2.61 4.47	-0.581 0.0642	-5.31 1.11	0.288 0.0131	0.0555 0.0204
Robust design with eigenvector constraint	$-3.629 \pm j5.14$ -3.00 -19.0 -19.5	-1.11 12.37	-2.01 -1.83	-3.12 10.54	-0.606 0.361	-5.72 5.97	0.291 -0.0239	0.0543 0.0342
Robust design with matrix Q	$-3.619 \pm j5.18$ -2.40 -19.0 -19.5	-0.890 9.90	-1.68 -5.76	-2.57 4.14	-0.552 -0.243	-5.29 0.925	0.259 0.331	0.0540 0.0363
Performance robust design	$-3.619 \pm j5.18$ -2.71 -19.0 -19.5	-1.01 11.14	-1.68 -5.67	-2.69 5.47	-0.557 -0.183	-5.31 1.23	0.262 0.290	0.0529 0.0476

and Shapiro[1] are for the outputs $y^T = [q, n_{zp}, \gamma, \delta_e, \delta_f]$. However, for simplicity, in this paper we will use full-state-feedback. The full state feedback gain matrix is shown in Table 2, and it is obtained by applying a transformation to the feedback gain matrix described by Sobel and Shapiro.[1] The achievable eigenvectors are shown in Table 3, where we observe that the two desired zero entries in the short-period eigenvectors have been achieved exactly and the desired zero entry in the gamma mode eigenvector is very small. Thus, we expect excellent decoupling in both the pitch pointing and vertical translation responses. Finally, feedforward gains based on O'Brien and Broussard's[9] command generator tracker are computed to obtain zero steady-state error to a step command. Thus, the control law is given by

$$u = [\Omega_{22} - F\Omega_{12}]u_c + Fy \tag{21}$$

where

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} = \begin{bmatrix} A & B \\ H & 0 \end{bmatrix}^{-1}$$

and

$$u_c = \begin{bmatrix} \gamma_c \\ \theta_c \end{bmatrix}$$

and where the matrix H is chosen so that

$$\begin{bmatrix} \gamma \\ \theta \end{bmatrix} = Hx$$

The control law described by Eq. (21) consists of a feedforward part to achieve zero steady-state error to a step command u_c and a feedback part to achieve the desired transient response. A more detailed description of the command generator tracker control law described by Eq. (21) is given in Ref. 1.

The vertical translation and pitch pointing responses are shown in Fig. 2. As expected, the decoupling is excellent for both cases. However, the design of Ref. 1 does not consider stability robustness when the aircraft is subject to linear time-varying structured state-space uncertainty.

Next, we consider a robust eigenstructure assignment pitch pointing/vertical translation control law. First, to reduce the

Table 3 Closed-loop eigenvectors (normalized $\|x\|_\infty = 1$)

	Short period		Gamma mode	Actuator mode	Actuator mode
Nonrobust design	0.0000 0.5954 -0.1339 -0.4502 1.0000	0.0000 -0.7679 0.0369 -0.2629 0.0000	0.3089 0.0001 -0.3090 -0.8656 1.0000	-0.0054 1.0000 -0.0472 0.9317 0.0081	-0.0135 0.0343 0.0118 -0.0249 1.0000
Robust design	-0.0434 1.0000 -0.0472 0.0215 0.2153	0.0301 0.0000 -0.1598 -0.7004 -0.0044	-0.2668 -0.0038 0.2682 0.5802 1.0000	-0.0052 1.0000 -0.0475 0.9316 -0.0060	-0.0137 0.0631 0.0104 0.0029 1.0000
Robust design eigenvector constraint	-0.0200 1.0000 -0.0716 -0.0685 0.5301	0.0077 0.0000 -0.1378 -0.7063 0.6895	-0.1911 0.0010 0.1908 0.3868 1.0000	-0.0053 1.0000 -0.0474 0.9312 -0.0018	-0.0137 0.0599 0.0106 -0.0002 1.0000
Robust design with matrix Q	-0.0418 1.0000 -0.0486 -0.0177 0.5985	0.0119 0.0000 -0.1419 -0.6719 0.1743	-0.3711 0.0008 0.3708 0.8377 1.0000	-0.0053 1.0000 -0.0473 0.9325 0.0018	-0.0137 0.0587 0.0106 -0.0013 1.0000
Performance robust design	-0.0421 1.0000 -0.0485 -0.0180 0.6023	0.0119 0.0000 -0.1420 -0.6714 0.1711	-0.2528 -0.0026 0.2537 0.5440 1.0000	-0.0053 1.0000 -0.0473 0.9374 0.0027	-0.0137 0.0599 0.0106 -0.0001 1.0000

conservatism in Eq. (8), we use a constant similarity transformation to obtain the transformed state given by

$$\tilde{x}(t) = [\theta, q, \alpha, \delta_e, \delta_f]^T \tag{22}$$

The utilization of such a transformation is valid because the stability of a linear time-varying system is preserved when subjected to a constant similarity transformation.[10] The use of such a similarity transformation to reduce conservatism was used by Yedavalli and Liang[11] in connection with a Lyapunov approach to robustness. They indicate that there does not yet exist a systematic method for choosing the similarity transformation. We choose the transformation for our example to reduce the magnitude of the entries in the uncertainty matrix

A_{max} with the conjecture that this will reduce the quantity on the right-hand side of Eq. (8). In particular, our choice of the transformation yields a transformed system with a matrix A_{max} whose first row is zero.

In the transformed coordinates, the equation $\dot{\theta} = q$ is a physical relationship without uncertain parameters. Thus, the first row of A_{max} will contain all zeros. The matrices A_{max} and B_{max} are chosen to be

$$A_{max} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0869 & 1.08 & 0.4313 & 0.158 \\ 0 & 0 & 0.1341 & 0.0169 & 0.0252 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.1 \end{bmatrix} \tag{23a}$$

$$B_{max} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \tag{23b}$$

This choice of A_{max} and B_{max} is for illustrative purposes. The A_{max} and B_{max} chosen correspond to maximum uncertainty of 10% in a_{22}, a_{25}, a_{33}, a_{34}, and a_{35}; 2.5% in a_{23} and a_{24}; and 0.5% in the actuator parameters a_{44}, a_{55}, b_{41}, and b_{52}.

We emphasize that the transformed system is used only for computation of Eq. (8), whereas the original system of Eq. (20) is used for eigenvalue/eigenvector selection and gain computation. For the design of Ref. 1, $\alpha = 1.0$, whereas the right-hand side of Eq. (8) equals 4.28, which indicates that the sufficient condition for robust stability is not satisfied. We use a constrained optimization to solve

$$\min \lambda_{max}\{(M^{-1})^{+}[A_{max} + B_{max}(FC)^{+}]M^{+}\} - \alpha \tag{24}$$

while constraining some of the eigenvalues and eigenvectors. First, we choose to assign the actuator eigenvalues and eigenvectors to the same values as in the design presented in Ref. 1. Then, an optimization is performed over a subset of the short-

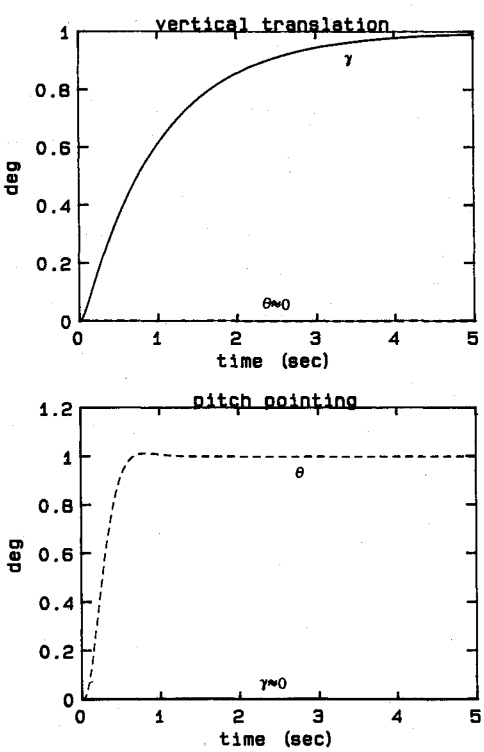

Fig. 2 Nonrobust design time responses.

period eigenvector parameters, short-period eigenvalues, and the gamma mode eigenvalue. The term "eigenvector parameters" is the vector z_i, which is defined in Eq. (15). This choice of optimization parameters is chosen by examining the gradient at the initial point corresponding to the design in Ref. 1. This gradient is given by

$$\text{grad} = \begin{bmatrix} -1.75 \\ -0.389 \times 10^{-7} \\ -0.767 \\ 0.393 \times 10^{-6} \\ -0.659 \\ -0.461 \\ -0.121 \\ -0.0525 \\ 1.47 \\ 0.390 \\ 0.109 \\ 0.140 \\ -0.0455 \\ 0.181 \\ 0.00567 \end{bmatrix} \quad \begin{matrix} \mathrm{Re}[z_{sp}(1)] \\ \mathrm{Re}[z_{sp}(2)] \\ \mathrm{Im}[z_{sp}(1)] \\ \mathrm{Im}[z_{sp}(2)] \\ \mathrm{Re}[\lambda_{sp}] \\ \mathrm{Im}[\lambda_{sp}] \\ z_{\gamma}(1) \\ z_{\gamma}(2) \\ \lambda_{\gamma} \\ z_{\delta_e}(1) \\ z_{\delta_e}(2) \\ \lambda_{\delta_e} \\ z_{\delta_f}(1) \\ z_{\delta_f}(2) \\ \lambda_{\delta_f} \end{matrix}$$

We choose to optimize over the parameters that correspond to $|g_i| > 0.46$. The parameter vector for the optimization is chosen to be

$$p = [\mathrm{Re}\, z_{sp}(1),\ \mathrm{Im}\, z_{sp}(1),\ \mathrm{Re}\,\lambda_{sp},\ \mathrm{Im}\,\lambda_{sp},\ \lambda_{\gamma}]^T$$

Thus, during the optimization, the first entry (both real and imaginary parts) of the vector z_{sp} for the short-period mode is allowed to change; the short-period eigenvalues are allowed to change subject to the constraints $-7.6 \le \mathrm{Re}[\lambda_{sp}] \le -3.6$, $3.2 \le \mathrm{Im}[\lambda_{sp}] \le 5.2$; and the gamma mode eigenvalue is allowed to change subject to the constraint $-2.65 \le \lambda_{\gamma} \le -0.5$. Each time λ_{γ} is changed, its eigenvector is computed as the orthogonal projection of the desired gamma mode eigenvector onto the current achievable gamma mode subspace. In this way, the zero entry in the gamma mode eigenvector might be achieved, in which case the vertical translation decoupling may be nearly the same as in the design shown in Ref. 1.

The constrained optimization is performed as a sequence of 10 unconstrained optimizations by using the IMSL subroutine ZXMIN,[12] which implements a variable metric method. The first unconstrained optimization is initialized at the design shown in Ref. 1, with the variable ϵ in Eq. (16) set equal to 0.1. The variable ϵ is reduced by a factor of 10 each time a new unconstrained minimization is performed.

When the optimization is complete, $\alpha = 2.65$ and the right-hand side of Eq. (8) equals 2.57; thus, the sufficient condition for robust stability is satisfied. The closed-loop eigenvectors are shown in Table 3, from which we observe that the zero entry in the gamma mode eigenvector has been achieved. This suggests that the vertical translation decoupling will be the same as in the design shown in Ref. 1. However, the entries in the short-period eigenvectors that were zero in the Ref. 1 design are now nonzero. This suggests that the pitch pointing decoupling will not be as good as in the Ref. 1 design. The closed-loop eigenvalues and feedback gains are shown in Table 2. Observe that the short-period eigenvalues have moved to the boundary of the constraint region corresponding to the largest settling time and smallest damping ratio. The gamma mode eigenvalue has moved to the leftmost point in its constraint interval, which is to be expected since λ_{γ} appears explicitly in Eq. (8). Also observe that the magnitude of the feedback gains has been reduced. The vertical translation and pitch pointing responses are shown in Fig. 3. As expected, the vertical translation decoupling is virtually identical to the Ref. 1 design, whereas the pitch pointing decoupling has been degraded.

The first column of Table 4 shows the maximum absolute value of gamma for the pitch pointing responses from which we can observe the degradation in the pitch pointing decoupling. Table 4 also shows that the new robust eigenstructure assignment design exhibits a significantly improved minimum singular value of the return difference matrix at the inputs and an improved condition number of the closed-loop modal matrix as compared to the Ref. 1 design. We also observe that the minimum singular value of the return difference matrix at the outputs has shown a slight improvement although it still remains small. For our example, we conclude that the robustness optimization yields a significant improvement in the multivariable stability margins at the aircraft inputs while having little effect on the margins at the aircraft outputs. Nevertheless, robust stability is guaranteed for the structured state-space uncertainty, which is represented by the bounds described by the matrices $A_{\max}$ and $B_{\max}$.

Next, we place a constraint on the first entry of the complex conjugate short-period eigenvectors. This will allow us to place additional emphasis on the mode decoupling that is required for the pitch pointing maneuver. We remark that, in the nonrobust design, we achieved mode decoupling by choosing each achievable eigenvector v_i^a to be close to its corresponding desired eigenvector v_i^d in a 2-norm sense. However, in the robust design, we optimize over the short-period eigenvector parameters instead of choosing v_{sp}^a to be close to v_{sp}^d in a 2-norm sense. Hence the need for additional eigenvector constraints when the designer chooses to emphasize mode decoupling. The constraints are chosen to be $|\mathrm{Re}v_{sp}(1)| \le 0.01$ and $|\mathrm{Im}v_{sp}(1)| \le 0.01$, where the eigenvectors are normalized to unit length in the 2-norm sense. A solution could not be obtained for the eigenvalue constraints that were used in the previous design. Therefore, the gamma mode eigenvalue constraint is relaxed to become $-3.0 \le \lambda_\gamma \le -0.5$, which will have the effect of moving the gamma mode eigenvalue farther into the left half of the complex plane. The new closed-loop eigenvectors are shown in Table 3, where we observe that the real and imaginary parts of the first entry of the short-period eigenvectors are significantly smaller than in the previous design. Thus, we should expect that the pitch pointing response will be improved. The vertical translation and pitch pointing responses are shown in Fig. 4. We observe that the coupling between γ and θ has been reduced by approximately 50% as compared with the previous design. However, we observe from Table 2 that the feedback gains are larger in magnitude. We also observe from Table 4 that we no longer have a significant improvement in the minimum singular value of the return difference matrix at the inputs. Therefore, we observe a tradeoff between stability robustness and pitch pointing performance.

Next, we remove the additional short-period eigenvector constraints, but we introduce a unitary weighting matrix Q to reduce the conservatism in the robust stability condition. In this case, the sufficient condition of Eq. (8) is replaced by the following[3]:

$$\alpha > \kappa_2(Q) \cdot \| [(MDQ)^{-1}]^+ [A_{\max} + B_{\max}(FC)^+][MDQ]^+ \|_2 \quad (25)$$

where

$$\alpha = -\max_i \mathrm{Re}\lambda_i(A + BFC)$$

and $\kappa_2(Q) = \|Q^{-1}\|_2 \cdot \|Q\|_2$ is the 2-norm condition number of the matrix Q.

The computation of the matrix D is based on the following result, which is due to Stoer and Witzgall[5]:

Lemma. Consider a nonnegative matrix $A^+ \in R^{n\times n}$ and let $\pi(A^+)$ denote the Perron eigenvalue of A^+. Denote by x and y the respective right and left Perron eigenvectors of A^+

Table 4 Comparison of robustness measures

	Max $\|\gamma\|$ $\theta_c = 1$	Min$\underline{\sigma}$ $(I + FG)$	Min$\underline{\sigma}$ $(I + GF)$	cond(M)
Nonrobust design	6.66×10^{-4}	0.1983	0.0491	56.00
Robust design	0.2171	0.5021	0.0563	27.30
Robust design eigenvector constraints	0.0787	0.2802	0.0408	36.80
Robust design with matrix Q	0.1617	0.4244	0.0516	27.11
Performance robust design	0.1596	0.4143	0.0504	27.84

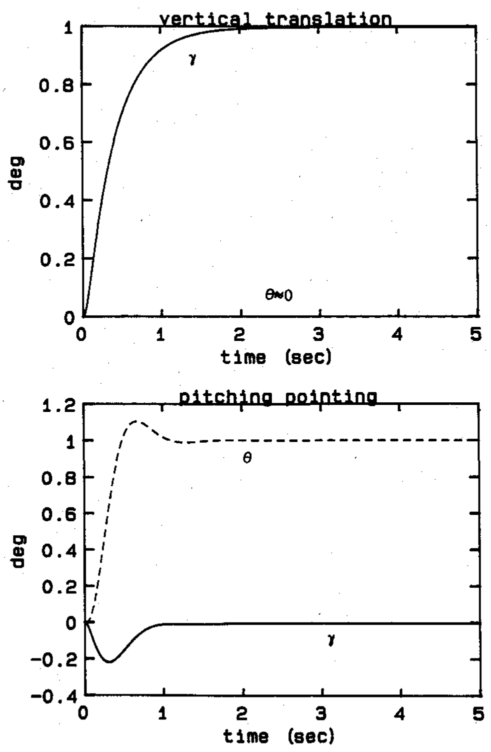

Fig. 3 Robust design time responses.

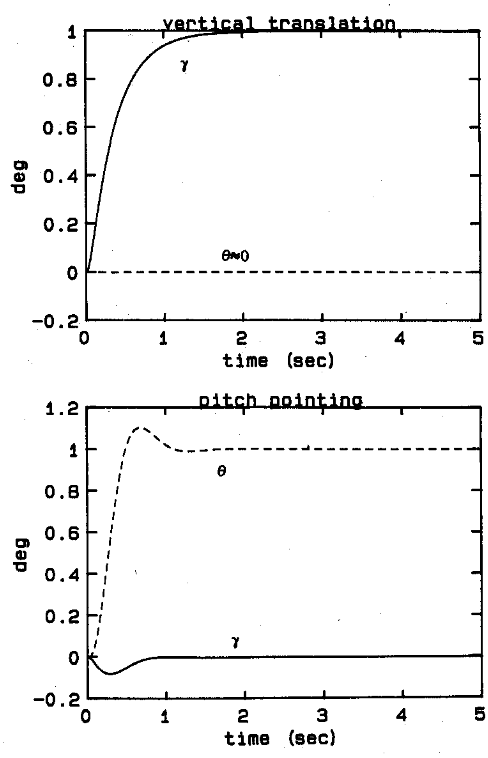

Fig. 4 Robust design with eigenvector constraint time responses.

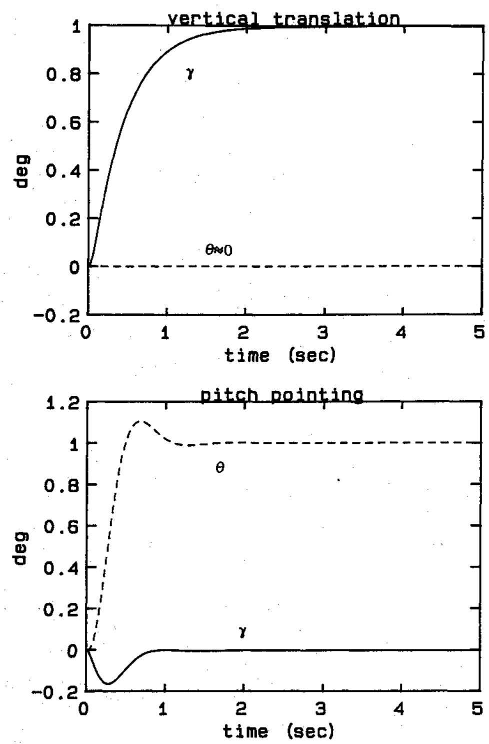

Fig. 5 Robust design with matrix Q time responses.

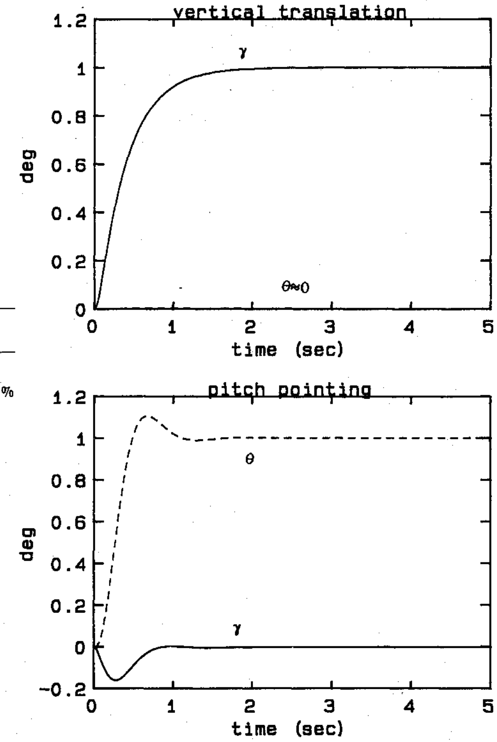

Fig. 6 Performance robust design time responses.

satisfying $\pi x = A^+x$ and $\pi y^T = y^TA^+$ normalized so that $\|x\|_\infty = \|y\|_\infty = 1$. Then,

$$\inf_D \|D^{-1}A^+D\|_2 = \pi(A^+) \tag{26}$$

where

$$D = \mathrm{diag}\{[y(1)/x(1)]^{1/2}, \ldots, [y(n)/x(n)]^{1/2}\} \tag{27}$$

Yu and Sobel[3] conjecture that a good choice for the matrix Q is given by

$$Q = UV^T \tag{28}$$

where $U\Sigma V^T$ is the singular-value decomposition of the matrix $(MD)^T$. The method for computing the matrix Q is first to obtain the matrix D, which is described by Eq. (27) where x and y are the right and left Perron eigenvectors of

$$[(MD)^{-1}]^+[A_{\max} + B_{\max}(FC)^+][MD]^+ \tag{29}$$

or, equivalently (because D is real, positive, and diagonal),

$$D^{-1}(M^{-1})^+[A_{\max} + B_{\max}(FC)^+](M)^+D \tag{30}$$

This will yield the optimal matrix D when Q is the identity matrix. Then, the matrix Q is computed by using Eq. (28). Thus, we utilize both a real, positive, diagonal weighting matrix D and a unitary weighting matrix Q.

By using matrix Q, we can tighten the constraint on the gamma eigenvalue to $-2.4 \le \lambda_\gamma \le -0.5$. Thus, we obtain a solution with $\lambda_\gamma = -2.4$, which is to be compared with the previous solution of $\lambda_\gamma = -2.65$. Thus we do not need to move the gamma mode eigenvalue as far left in the complex plane as before. The vertical translation and pitch pointing responses are shown in Fig. 5. The vertical translation response exhibits slightly less coupling than the first robust design of Fig. 3, which did not utilize either the unitary matrix Q or the explicit eigenvector constraints on $|\mathrm{Re}\,v_{sp}(1)|$ and $|\mathrm{Im}\,v_{sp}(1)|$.

Finally, we design a controller by optimizing the performance robustness condition of Eq. (10). We choose $a = -0.25$ and $\theta = 8.63$ deg, which corresponds to a region in Fig. 1 with $\zeta \ge 0.15$ and $t_s \le 4$ s. The quantity to be minimized is given by

$$\kappa_2(Q) \cdot \|[(MDQ)^{-1}]^+[A_{\max} + B_{\max}(FC)^+][MDQ]^+\|_2 - \min(\alpha_1,\alpha_2) \tag{31}$$

where

$$\alpha_1 = -\max_i\,[\mathrm{Re}\lambda_i] - 0.25$$

and

$$\alpha_2 = -\max_i\,[\mathrm{Re}\,e^{-j(8.63)(\pi/180)}\lambda_i]$$

For the design of Ref. 1, we find that $\min(\alpha_1, \alpha_2) = 0.75$ whereas the right-hand side of Eq. (10) equals 4.546. Thus, the sufficient condition for performance robustness is not satisfied. The constraint for the gamma mode eigenvalue is chosen to be $-2.70 \le \lambda_\gamma \le -0.5$. After optimizing Eq. (31), we find that $\min(\alpha_1, \alpha_2) = 2.46$ while the right hand side of Eq. (10) equals 2.393. Thus, the sufficient condition for performance robustness is satisfied. The vertical translation and pitch pointing responses are shown in Fig. 6. We observe that these responses are almost identical to the robust stability responses shown in Fig. 5. However, we now guarantee that the closed-loop eigenvalues remain in the performance region for all time-invariant uncertainty that satisfies the $A_{\max}$ and $B_{\max}$ bounds. We observe from Table 4 that this design has the smallest coupling of all the robust designs that do not use constraints on the first entry of the complex conjugate short-period eigenvectors.

Conclusion

Robust eigenstructure assignment has been utilized to design a pitch pointing/vertical translation control law for a high-performance fighter aircraft. The example shows the feasibility of optimizing a sufficient condition for either robust stability or robust performance for an aircraft with structured state-space uncertainty. The conservatism of the robustness sufficient conditions may be reduced by introducing a similarity transformation and two weighting matrices into the design method.

The example illustrates the tradeoff between robustness and nominal pitch pointing performance especially in the coupling between the pitch attitude command and the flight-path-angle transient response. For the chosen example, the robust eigenstructure assignment design exhibits both an improved minimum singular value of the return difference matrix at the inputs and an improved condition number of the closed-loop modal matrix.

Acknowledgments

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